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The principle of the indistinguishability of identical particles and the Lie algebraic approach to the field quantisation

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Abstract. The density matrix, rather than the wavefunction describing the system of a fixed number of non-relativistic identical particles, is subject to the second quantisation. Here the bilinear operators which move a particle from a given state to another appear and satisfy the Lie algebraic relations of the unitary group SU(p) when the dimension $p \to \infty$. The drawing into consideration of the system with a variable number of particles implies the extension of this algebra into one of the simple Lie algebras of classical (orthogonal, symplectic or unitary) groups in the even-dimensional spaces. These Lie algebras correspond to the para-Fermi-, para-Bose- and para-uniquantisation of fields, respectively.

1. Introduction

Nowadays it is widely accepted that all known elementary and composite particles of nature and even hypothetical ones (quarks, gluons when colour degrees of freedom are taken into account) are necessarily either fermions or bosons described by antisymmetric and symmetric wavefunctions, respectively. However, right from the start of quantum mechanics its founders drew attention to the possible existence of some kind of 'intermediate statistics' of identical particles (for example, see Dirac 1958, Pauli 1958). These statistics could be described by the functions belonging to the multidimensional irreducible representations of the group of permutations of particle indices. Herein the state of a system of identical particles corresponds to a 'generalised ray' in the Hilbert space of many-particle states: the set of normalised vectors in such irreducible subspaces (Messiah and Greenberg 1964). The physics literature contains a number of attempts to disprove this possibility (Galindo *et al* 1962, Pandres 1962, Steinmann 1966) but all of them turn out to be unsound because the arguments when formulated correctly (Dresden 1963, Messiah and Greenberg 1964, Hartle and Taylor 1969, Stolt and Taylor 1970, 1971) always imply an additional assumption. Many authors (among them Salzmann (1970) and Kaplan (1974)) proposed manifestly more restrictive formulations of the law of the indistinguishability of identical particles, which were equivalent to 'the symmetrisation postulate' (Messiah and Greenberg 1964): the existence of the Fermi–Dirac and Bose–Einstein statistics only.

From the foregoing discussion we conclude that there are no reasons for rejecting intermediate statistics, or 'parastatistics' as they were named later, from the outset. Thus we ought to investigate the question: what kind of matter are they?

It is well known that the most convenient way to describe systems of identical particles is the method of second quantisation, as the particle creation and annihilation operators contain full information about the permutation properties of the corresponding wavefunctions. Thus one should use this method for wavefunctions with mixing symmetry. For the first time an attempt of this kind was undertaken by Okayama (1952), but later Kamefuchi and Takahashi (1962) showed that there was a mistake in his consideration. The analogous attempt was not repeated.

Here we apply another method. Rather than the wavefunction the density matrix is subject to the second quantisation because the latter is 'an observable' and must be a symmetrical function of particle indices due to the indistinguishability of identical particles. In the framework of this approach bilinear operators which move a particle from a given state to another appear and come into the Schrödinger equation for the density matrix. The main result is that they obey the Lie algebra of the unitary group $SU(p)$ when the dimension p, equal to the number of independent single-particle states, goes to infinity. Thus any generalisation of the usual field quantisation must include the algebra of those operators.

At first we consider the non-relativistic theory, but the relativistic generalisation can be easily accomplished after the transition from the algebra of bilinear operators to the generalised algebra of particle creation and annihilation operators is performed.

The paper is organised as follows. In § 2 the second quantisation of the density matrix is carried out and the Schrödinger equation is written down in terms of bilinear operators; in the same section the algebra of these operators is considered.

In § 3 the transition to systems with a variable number of particles is performed and the particle creation and annihilation operators are introduced in themselves. Also the algebra of bilinear operators is expanded into the Lie algebra of one of the classical (orthogonal, symplectic, unitary) groups. In § 4 the connection between this generalisation of field quantisation and fermion and boson fields with inner degrees of freedom is briefly discussed.

2. The second quantisation of the denisty matrix

The system of n identical particles is described unambiguously by the density matrix

$$\rho(x_1, \ldots, x_n; x'_1, \ldots, x'_n; t) \tag{1}$$

which is a complex function of time, t, and two sets of the particle arguments: 'primary' $x_1, \ldots, x_n$ and 'secondary' $x'_1, \ldots, x'_n$ within the $3n$-dimensional configuration space. (If arguments include, in addition to the space coordinates, the spin variables, then the dimension of the configuration space enlarges.) Beforehand we do not assume any symmetry properties of this density matrix with respect to permutations of primary arguments or permutations of secondary arguments performed separately. However, owing to the indistinguishability of identical particles, the density matrix (1), being an observable, must be symmetrical with respect to the same permutation performed among its primary and secondary arguments simultaneously. Thus

$$\rho(x_{P1}, \ldots, x_{Pn}; x'_{P1}, \ldots, x'_{Pn}; t) = \rho(x_1, \ldots, x_n; x'_1, \ldots, x'_n; t) \tag{2}$$

for any permutation $P \in S_n$, where P1, P2, Pn means the replacement of particle indices $1, 2, \ldots, n$ by some other indices $i_1, i_2, \ldots, i_n$ from the same set of numbers. We consider this rule (2) as the most general formulation of the principle of indistinguishability of identical particles. Our task now is to derive the consequences of this law.

In order to approach second quantisation, it is necessary to introduce the space of the occupation numbers. Let there be a complete set of functions $\phi^{(r)}(x)$ which are eigenvectors of the complete set of single-particle observables with the (discrete) eigenvalues denoted by (r). We decompose the density matrix into the series

$$\begin{aligned}\rho(x_1, \ldots, x_n; x_1', \ldots, x_n'; t) &= \sum_{\substack{r_1,\ldots,r_n \\ r_1',\ldots,r_n'}} c(r_1, \ldots, r_n; r_1', \ldots, r_n'; t) \\ &\quad \times \overline{\phi^{(r_1')}}(x_1')\phi^{(r_1)}(x_1) \ldots \overline{\phi^{(r_n')}}(x_n')\phi^{(r_n)}(x_n)\end{aligned} \tag{3}$$

where the sum is taken over all sets of primary and secondary eigenvalues of n-particle states; the bar denotes complex conjugation.

The density matrix satisfies the following well-known properties (see, for example, Landau and Lifshitz 1958, Fano 1957). It is an Hermitian, positive definite, normalised to unity,

$$\sum_{r_1,\ldots,r_n} c(r_1, \ldots, r_n; r_1, \ldots, r_n; t) = 1, \tag{4}$$

matrix, which obeys the Schrödinger equation

$$i\hbar\, \partial c(t)/\partial t = Hc(t) - c(t)H. \tag{5}$$

Here $H(r_1, \ldots, r_n; r_1', \ldots, r_n')$ is the Hamiltonian of the system of identical particles in the r representation which must be symmetrical in its arguments.

Due to the absence of symmetry properties of the density matrix with respect to its primary or secondary arguments separately we cannot introduce any occupation numbers for primary-particle states and for secondary ones. However, we can do this for primary and secondary-particle states together. We define the number of particles n_{ij}, which is the number of identical particles occupying a given state $r^{(i)}$ among all primary states $r_1, \ldots, r_n$ and a given state $r^{(j)}$ among all secondary states $r_1', \ldots, r_n'$. In one word, n_{ij} is the number of identical particles in 'a double state' $(r^{(i)}, r^{(j)})$. Thus we shall speak about 'the double occupation number space'. Due to the symmetry of the density matrix in particle indices $1, 2, \ldots, n$ the matrices with the same double occupation numbers are equal to each other, and they can be put equal to a single density matrix in this space:

$$c(r_1, \ldots, r_n; r_1', \ldots, r_n'; t) = R(n_{11}, n_{12}, \ldots, n_{21}, n_{22}, \ldots; t) \tag{6}$$

$$\sum_{i,j=1}^{\infty} n_{ij} = n. \tag{7}$$

In a shorthand form we shall denote the set of numbers (n_{11}, n_{12}, etc) by the symbol (n_{ij}) and write $R(n_{ij})$ instead of the RHS of equation (6).

The single diagonal matrix

$$R(n_{ii}; t) \equiv R(n_{ii}, i = 1, \ldots, \infty; n_{ij} = 0 \text{ for } i \neq j; t)$$

in the double occupation number space corresponds to the set of diagonal matrices

$c(r_1, \ldots, r_n; r'_1, \ldots, r'_n; t)$ with the same double occupation states. The number of these matrices is equal to

$$n!\left(\prod_{i=1}^{\infty} n_{ii}!\right)^{-1}.$$

Thus the normalised density matrix in the double occupation number space is

$$f(n_{ii}; t) = \left(n! \bigg/ \prod_{i=1}^{\infty} n_{ii}!\right) R(n_{ii}; t) \tag{8}$$

$$\sum_{n_{ii}=1}^{\infty} f(n_{ii}; t) = 1. \tag{9}$$

It is just the normalised density matrix (8) that has the meaning of the probability of finding the number n_{11} of identical particles in the state $r^{(1)}$, the number n_{22} of identical particles in the state $r^{(2)}$, etc. Now we define the general normalised density matrix in the form

$$f(n_{ij}; t) = n!\left(\prod_{i,j=1}^{\infty} n_{ij}!\, n_{ji}!\right)^{-1/2} R(n_{ij}; t). \tag{10}$$

The properties of the density matrix in the double occupation number space can be written as follows.

(i) Hermiticity condition

$$\bar{f}(n_{ij}; t) = f(n_{ji}; t). \tag{11}$$

(ii) Positive definiteness: positiveness of the diagonal matrix elements

$$f(n_{ii}, i = 1, \ldots, \infty; n_{ij} = 0 \text{ for } i \neq j; t) \geq 0 \tag{12a}$$

and the Cauchy inequality

$$f(N_{ii}; t) f(N'_{jj}; t) \geq |f(n_{ij}; t)|^2 \tag{12b}$$

where the arguments N_{ii} and N'_{jj} of the diagonal matrix elements in the LHS of inequality (12*b*) are equal to the total numbers of particles in the primary i and secondary j states, respectively:

$$N_{ii} = \sum_{j=1}^{\infty} n_{ij} \qquad N'_{jj} = \sum_{i=1}^{\infty} n_{ij}. \tag{13}$$

We remark here that in the cases of ordinary Fermi–Dirac and Bose–Einstein statistics there is the degeneration

$$f(n_{ij}; t) = \bar{\chi}(N'_j; t)\chi(N_i; t). \tag{14}$$

In these cases the numbers n_{ij} are formed by the following procedure. The intervals of length N_1, N_2, etc, are put one by one on the upper edge of the number axis, and the intervals of lengths N'_1, N'_2, etc, are put analogously on the lower edge of the same axis. Then the intervals between any two neighbouring bounds (upper and lower together) give the double occupation numbers n_{11}, n_{12}, etc. Obviously, arbitrary transpositions of the sequences of upper intervals as well as lower intervals are allowed. By transposition of this kind we get new double occupation numbers m_{11}, m_{12}, etc, and the new density matrix must be equal to the old one:

$$f(n_{ij}; t) = f(m_{ij}; t). \tag{15}$$

In this case the condition (12*b*) turns into the equality. Conversely, the existence of such a kind of degeneration (15) implies the representation of the density matrix in the form (14) and therefore the availability of the usual Fermi–Dirac or Bose–Einstein statistics (of course, in the first case the numbers N_i, N'_j must be ≤ 1). In the considered general case the degeneration (15) does not hold; hence the knowledge of the numbers N_i, N'_j from (13) cannot unambiguously restore the density matrix in the double occupation number space.

We can now write down the Schrödinger equation in terms of the double occupation numbers. For definiteness we consider a system of particles with the pairing interaction $G(x, y)$ in the external potential $V(x)$. For this system the Schrödinger equation is

$$\begin{aligned} \mathrm{i}\hbar\, \partial f(n_{ij}; t)/\partial t &= \sum_{k=1}^{\infty} \epsilon^{(k)} \sum_{p=1}^{\infty} (n_{kp} - n_{pk}) f(n_{ij}; t) \\ &+ \sum_{k,p=1}^{\infty} \sum_{r=1}^{\infty} [V_{k;r}[n_{kp}(n_{rp}+1)]^{1/2} f(n_{kp}-1, n_{rp}+1; t) \\ &- \bar{V}_{k;r}[n_{pk}(n_{pr}+1)]^{1/2} f(n_{pk}-1, n_{pr}+1; t)] \\ &+ \tfrac{1}{2} \sum_{k,l=1}^{\infty} \sum_{p,q=1}^{\infty} \sum_{r,s=1}^{\infty} [G_{kl;rs}[n_{kp}n_{lq}(n_{rp}+1)(n_{sq}+1)]^{1/2} \\ &\times f(n_{kp}-1, n_{lq}-1, n_{rp}+1, n_{sq}+1; t) \\ &- \bar{G}_{kl;rs}[n_{pk}n_{ql}(n_{pr}+1)(n_{qs}+1)]^{1/2} \\ &\times f(n_{pk}-1, n_{ql}-1, n_{pr}+1, n_{qs}+1; t)] \end{aligned} \tag{16}$$

where $\epsilon^{(k)}$ is an eigenvalue of a single-particle energy in the state $r^{(k)}$ and

$$V_{k;r} = \int \overline{\phi^{(k)}}(x) V(x) \phi^{(r)}(x)\, \mathrm{d}x \tag{17}$$

$$G_{kl;rs} = \int \overline{\phi^{(k)}}(x) \overline{\phi^{(l)}}(y) G(x, y) \phi^{(r)}(x) \phi^{(s)}(y)\, \mathrm{d}x\, \mathrm{d}y. \tag{18}$$

In the RHS of equation (16) only the varying arguments are indicated.

The double occupation number space is the Hilbert space with the scalar product of vectors

$$\langle f|g\rangle = \sum_{n_{ij}=1}^{\infty} \bar{f}(n_{ij}) g(n_{ij}) = \overline{\langle g|f\rangle}. \tag{19}$$

Let us emphasise that the vectors in this space are the density matrices themselves but not wavefunctions.

In this space one can introduce the basis vectors with fixed double occupation numbers

$$|n_{ij}^0\rangle = \prod_{i,j=1}^{\infty} \delta_{n_{ij} n_{ij}^0} \tag{20}$$

where δ_{nm} is Kronecker's delta.

Obviously, there is the orthonormalisation

$$\langle m_{ij}^0 | n_{ij}^0 \rangle = \prod_{i,j=1}^{\infty} \delta_{m_{ij}^0 n_{ij}^0}. \tag{21}$$

Now the density matrix can be represented by the decomposition

$$|f(n_{ij};\, t)\rangle = \sum_{n_{ij}^0=1}^{\infty} f(n_{ij}^0;\, t)|n_{ij}^0\rangle. \tag{22}$$

Each coefficient of the decomposition (22) is the value of the density matrix at 'the point $n_{ij} = n_{ij}^0$'

$$\langle n_{ij}^0|f(n_{ij};\, t)\rangle = f(n_{ij}^0;\, t). \tag{23}$$

We are now in a position to introduce the operators of the transition of a particle from one primary state, say s, into another primary state, say r, without any transitions of secondary-particle states. We define these operators by their operating on the basis vectors:

$$N_{rs}|n_{ij}^0\rangle = \sum_{q=1}^{\infty} [n_{sq}^0(n_{rq}^0 - \delta_{rs} + 1)]^{1/2}|n_{rq}^0 + 1,\, n_{sq}^0 - 1\rangle. \tag{24}$$

With this definition the operator N_{rr} is merely an operator of particle numbers in the primary state r:

$$N_{rr}|n_{ij}^0\rangle = \left(\sum_{q=1}^{\infty} n_{rq}\right)|n_{ij}^0\rangle. \tag{25}$$

The non-vanishing matrix elements are

$$\langle n_{rq}^0 + 1,\, n_{sq}^0 - 1|N_{rs}|n_{rq}^0,\, n_{sq}^0\rangle = [n_{sq}^0(n_{rq}^0 + 1)]^{1/2} \qquad \text{for } r \neq s \tag{26a}$$

$$\langle n_{ij}^0|N_{rr}|n_{ij}^0\rangle = \sum_{q=1}^{\infty} n_{rq}^0. \tag{26b}$$

Due to (24) and (20) the action of the operator N_{rs} onto the density matrix via (22) gives

$$N_{rs}f(n_{ij};\, t) = \sum_{q=1}^{\infty} [(n_{sq} - \delta_{rs} + 1)n_{rq}]^{1/2} f(n_{rq} - 1,\, n_{sq} + 1;\, t). \tag{27}$$

We remark that if there is a product of two or several operators, say $N_{rs}N_{lm}$, then they operate on the density matrix beginning from the leftmost one of them, i.e. N_{rs}, and so on, in contrast to the order of their action on the basis vector.

In an analogous way we can introduce the operators which change the secondary states without changing the primary-particle states. We indicate them by a prime:

$$N'_{rs}|n_{ij}^0\rangle = \sum_{q=1}^{\infty} [n_{qs}^0(n_{qr}^0 - \delta_{rs} + 1)]^{1/2}|n_{qr}^0 + 1,\, n_{qs}^0 - 1\rangle \tag{28}$$

$$\langle n_{qr}^0 + 1,\, n_{qs}^0 - 1|N'_{rs}|n_{qr}^0,\, n_{qs}^0\rangle = [n_{qs}^0(n_{qr}^0 + 1)]^{1/2} \qquad \text{for } r \neq s \tag{29a}$$

$$\langle n_{ij}^0|N'_{ss}|n_{ij}^0\rangle = \sum_{q=1}^{\infty} n_{qs}^0 \tag{29b}$$

$$N'_{rs}f(n_{ij};\, t) = \sum_{q=1}^{\infty} [(n_{qs} - \delta_{rs} + 1)n_{qr}]^{1/2} f(n_{qr} - 1,\, n_{qs} + 1;\, t). \tag{30}$$

The matrix elements of Hermitian conjugate operators are defined in the usual way:

$$\langle m_{ij}^0|N_{rs}^+|n_{ij}^0\rangle = \overline{\langle n_{ij}^0|N_{rs}|m_{ij}^0\rangle} \tag{31a}$$

$$\langle m_{ij}^0|(N'_{rs})^+|n_{ij}^0\rangle = \overline{\langle n_{ij}^0|N'_{rs}|m_{ij}^0\rangle}. \tag{31b}$$

By means of (26) and (29) it is easy to prove that

$$N_{rs}^{+}=N_{sr} \qquad (N'_{rs})^{+}=N'_{sr}. \tag{32}$$

Thus the operators N_{rr} and N'_{rr} are self-adjoint:

$$N_{rr}^{+}=N_{rr} \qquad (N')_{rr}^{+}=N'_{rr}. \tag{33}$$

Now the Schrödinger equation (16) can be rewritten in terms of bilinear operators (27) and (30):

$$\begin{aligned} i\hbar\, \partial f(n_{ij};\, t)/\partial t &= \Big\{\sum_{k=1}^{\infty}\epsilon^{(k)}(N_{kk}-N'_{kk})+\sum_{k,r=1}^{\infty}(V_{k;r}N_{kr}-\bar{V}_{k;r}N'_{kr}) \\ &\quad+\tfrac{1}{4}\sum_{k,l=1}^{\infty}\sum_{r,s=1}^{\infty}[G_{kl;rs}(\{N_{kr}, N_{ls}\}_{+}-\delta_{rl}N_{ks}-\delta_{ks}N_{lr}) \\ &\quad-\bar{G}_{kl;rs}(\{N'_{kr}, N'_{ls}\}_{+}-\delta_{rl}N'_{ks}-\delta_{ks}N'_{lr})]\Big\}f(n_{ij};\, t). \end{aligned} \tag{34}$$

Here we have used the symmetric form $\{A, B\}_{+}\equiv AB+BA$ taking into account equation (40) obtained below.

It is convenient to apply a special differential representation for operators (27), (30). We can introduce the generating function

$$\mathcal{P}(z_{ij};\, t)=\sum_{n_{ij}=1}^{\infty}\prod_{i,j=1}^{\infty} z_{ij}^{n_{ij}}f(n_{ij};\, t) \tag{35}$$

where z_{ij} are real auxiliary variables: $0\leqslant z_{ij}\leqslant 1$. Conversely, the density matrix is expressed by

$$f(n_{ij};\, t)=\prod_{i,j=1}^{\infty}\left(\frac{1}{n_{ij}!}\frac{\partial^{n_{ij}}}{\partial z_{ij}^{n_{ij}}}\right)\mathcal{P}(z_{ij};\, t)\Big|_{\text{at } z_{ij}=0}. \tag{36}$$

The Hermiticity condition and the positive definiteness can be rewritten as

$$\bar{\mathcal{P}}(z_{ij})=\mathcal{P}(z_{ji}) \tag{37}$$

$$\mathcal{P}(z_{ii}, i=1,\ldots,\infty;\, z_{ij}=1 \text{ for } i\neq j;\, t)\geqslant 0. \tag{38}$$

The Schrödinger equation for the generating function has the same form (34) provided the operators (27) and (30) are given by the differentiation

$$N_{rs}=\sum_{q=1}^{\infty} z_{rq}\,\partial/\partial z_{sq} \qquad N'_{rs}=\sum_{q=1}^{\infty} z_{qr}\,\partial/\partial z_{qs}. \tag{39}$$

Now, directly from the definitions (27) and (30), or more easily with the help of the representation (39), one can derive the commutators of N_{rs}, N'_{ij}:

$$[N_{ij}, N_{rs}]=\delta_{jr}N_{is}-\delta_{is}N_{rj} \tag{40a}$$

$$[N'_{ij}, N'_{rs}]=\delta_{jr}N'_{is}-\delta_{is}N'_{rj} \tag{40b}$$

$$[N_{ij}, N'_{rs}]=0. \tag{41}$$

In the algebra (40) and (41) we immediately recognise the Lie algebra of the direct product of two unitary groups $SU(p)\times SU(p)'$ when the dimension p is equal to the number of independent single-particle states and goes to infinity.

3. The algebra of particle creation and annihilation operators

Hitherto we have considered only the systems with fixed numbers of (non-relativistic) identical particles. If we now wish to release this constraint, then we must introduce operators which vary the number of particles in the system. To this end we try 'to split' the operator N_{ij} into the product of creation b_i^+ and annihilation b_j operators. We suppose that N_{ij} has the bilinear form of commutator or anticommutator:

$$N_{ij} = \tfrac{1}{2}[b_i^+, b_j]_\epsilon + \text{constant} \qquad \epsilon = \pm \tag{42}$$

where $[A, B]_\epsilon \equiv AB + \epsilon BA$ (merely $[AB]$ indicates the commutator $AB - BA$). At present we have no profound grounds for this hypothesis (42). However, later we shall present some reasonable arguments in favour of this suggestion. Now we substitute the expressions (42) into equation (40a). We get

$$\underline{[b_i^+, [b_j, [b_r^+, b_s]_\epsilon]]_\epsilon} + \epsilon[b_j, [b_i^+, [b_r^+, b_s]_\epsilon]]_\epsilon = \underline{2\delta_{jr}[b_i^+, b_s]_\epsilon} - 2\delta_{is}[b_r^+, b_j]_\epsilon. \tag{43}$$

Here we used the identity

$$[[A, B]_\epsilon, [C, D]_\eta] = [A, [B, [C, D]_\eta]]_\epsilon + \epsilon[B, [A, [C, D]_\eta]]_\epsilon. \tag{44}$$

The comparison of the corresponding, e.g. underlined, terms in the LHS and RHS of equation (43) leads to the natural conjecture

$$[b_j, [b_r^+, b_s]_\epsilon] = 2\delta_{jr} b_s. \tag{45}$$

Obviously, equation (45), together with its Hermitian conjugate equation, is the solution of equation (43).

In addition to equation (45) we ought to postulate the additional relations

$$[b_j, [b_r, b_s]_\epsilon] = 0. \tag{46}$$

(In reality this relation is fulfilled automatically in the Fock representation with positive definite norms of state vectors and with vacuum or vacuum-like states.)

The relations (45) and (46) determine the algebra of operators b_i, b_j^+ completely. The other relations can be obtained from those either by means of the Hermitian conjugation or by applying the generalised Jacobi identity

$$[A, [B, C]_\epsilon] = -\epsilon[B, [A, C]_\epsilon] - [C, [A, B]_\epsilon]. \tag{47}$$

All relations can be represented in the unified form

$$[b(r), [b(s), b(t)]_\epsilon] = 2\omega(rs)b(t) + 2\epsilon\omega(rt)b(s) \tag{48}$$

in which any quantity $b(x)$ means b_x or b_x^+ denoted by $b^x \equiv b_x^+$; the symbols $\omega(xy)$ in the RHS of equation (48) have the following values:

$$\omega_{xy} = \omega^{xy} = 0 \qquad \omega_x{}^y = -\epsilon\omega^x{}_y = \delta_{xy}. \tag{49}$$

In the relations (48) we recognise the so-called 'Green paracommutation relations' postulated by Green (1953) in his generalisation of the field quantisation. The cases $\epsilon = +$ and $\epsilon = -$ correspond to para-Bose- and para-Fermi-quantisation, respectively.

Kamefuchi and Takahashi (1962) demonstrated that those relations are intimately connected with the Lie algebras of symplectic Sp($2p$) and orthogonal SO($2p$) groups in the even-dimensional spaces. Indeed, if, in addition to N_{ij} from (42), we also introduce the operators

$$M_{ij} = \tfrac{1}{2}[b_i, b_j]_\epsilon \qquad L_{ij} = \tfrac{1}{2}[b_i^+, b_j^+]_\epsilon = M_{ji}^+ \tag{50}$$

then it is easy to prove by equation (48) and the identity (44) at $\eta = \epsilon$ that the operators N_{ij}, M_{ij}, L_{ij} satisfy the Lie algebras of Sp($2p$) (for $\epsilon = +$) and SO($2p$) (for $\epsilon = -$) groups. Ryan and Sudarshan (1963) indicated that if one considers the group SO($2p+1$), then the operators b_i, b_j^+ themselves can be included in the Lie algebra of this group (of course, for $\epsilon = -$).

Thus we gained some mathematical reasons for our hypothesis (42) on the representation of the operator N_{ij} in the form of commutator or anticommutator only. It led us to simple Lie algebras. However, it is very promising to see that there is some physical foundation for this hypothesis also as an extension of Pauli's theorem on the spin-statistics connection to the parastatistics case (Dell'Antonio *et al* 1964).

Now the question arises: is it possible to correlate each simple Lie algebra to any scheme of the generalised quantisation? The answer is positive.

Obviously, the exceptional simple Lie algebras are not suitable for our aim to quantise the fields, the systems with the infinite number of degrees of freedom, because these algebras have finite and fixed dimensions (see, for example, Jacobson 1961). Thus there is just one more simple Lie algebra suitable for field quantisation: the Lie algebra of the unitary group SU($2p$) (or SU($2p+1$)). The corresponding quantisation has recently been developed by the author (Govorkov 1978) and Palev (1978, 1979). We shall call this scheme of quantisation 'the uniquantisation'. It can be constructed in such a manner that either para-Bose- or para-Fermi-quantisation will be its subalgebra.

For uniquantisation, in addition to the operators obeying (48), one introduces another set of particle creation and annihilation operators c_r, $c^{r} \equiv c_r^+$ obeying the same commutation relations

$$[c(r), [c(s), c(t)]_\epsilon] = 2\omega(rs)c(t) + 2\epsilon\omega(rt)c(s). \tag{51}$$

The following mutual commutation relations between b and c operators should be supposed:

$$[b(r), [b(s), c(t)]_\epsilon] = 4\omega(st)c(r) + 2\omega(rs)c(t) - 2\epsilon\omega(rt)c(s) \tag{52a}$$

$$[c(r), [c(s), b(t)]_\epsilon] = 4\omega(st)b(r) + 2\omega(rs)b(t) - 2\epsilon\omega(rt)b(s) \tag{52b}$$

$$[b(r), c(s)]_\epsilon = -[c(r), b(s)]_\epsilon \tag{52c}$$

$$[b(r), b(s)]_\epsilon = [c(r), c(s)]_\epsilon. \tag{52d}$$

We emphasise that because of (52d) the operators N_{ij} from (42) and M_{ij}, L_{ij} from (50) in our ansatz can be expressed in terms of the c_i^+, c_j operators on equal footing with the b_i^+, b_j operators.

Now we can introduce the complete set of operators N_{ij}, M_{ij}, L_{ij} from (42) and (50) and

$$\tilde{N}_{ij} = \tfrac{1}{2}[b_i^+, c_j]_\epsilon = -\tfrac{1}{2}[c_i^+, b_j]_\epsilon = -\tilde{N}_{ji}^+ \tag{53a}$$

$$\tilde{M}_{ij} = \tfrac{1}{2}[b_i, c_j]_\epsilon = -\tfrac{1}{2}[c_i, b_j]_\epsilon = -\epsilon\tilde{M}_{ji} \tag{53b}$$

$$\tilde{L}_{ij} = \tfrac{1}{2}[b_i^+, c_j^+]_\epsilon = -\tfrac{1}{2}[c_i^+, b_j^+]_\epsilon = -\epsilon\tilde{L}_{ji} = -\tilde{M}_{ji}^+. \tag{53c}$$

The commutators of those operators are

$$[N_{ij}, N_{rs}] = -[\tilde{N}_{ij}, \tilde{N}_{rs}] = \delta_{jr}N_{is} - \delta_{is}N_{rj} \tag{54a}$$

$$[M_{ij}, N_{rs}] = \delta_{jr}M_{is} + \epsilon\delta_{ir}M_{js} \tag{54b}$$

$$[L_{ij}, N_{rs}] = -\delta_{js}L_{ir} - \epsilon\delta_{is}L_{jr} \tag{54c}$$

$$[L_{ij}, M_{rs}] = -\epsilon\delta_{jr}N_{is} - \delta_{js}N_{ir} - \delta_{ir}N_{js} - \epsilon\delta_{is}N_{jr} \tag{54d}$$

$$[L_{ij}, L_{rs}] = [M_{ij}, M_{rs}] = 0 \tag{54e}$$

$$[\tilde{M}_{ij}, \tilde{N}_{rs}] = -\delta_{jr}M_{is} + \epsilon\delta_{ir}M_{js} \tag{54f}$$

$$[\tilde{L}_{ij}, \tilde{N}_{rs}] = -\delta_{js}L_{ir} + \epsilon\delta_{is}L_{jr} \tag{54g}$$

$$[\tilde{L}_{ij}, \tilde{M}_{rs}] = \epsilon\delta_{jr}N_{is} - \delta_{js}N_{ir} - \delta_{ir}N_{js} + \epsilon\delta_{is}N_{jr} \tag{54h}$$

$$[N_{ij}, \tilde{N}_{rs}] = \delta_{jr}\tilde{N}_{is} - \delta_{is}\tilde{N}_{rj} \tag{54i}$$

$$[\tilde{M}_{ij}, N_{rs}] = \delta_{jr}\tilde{M}_{is} - \epsilon\delta_{ir}\tilde{M}_{js} \tag{54j}$$

$$[\tilde{L}_{ij}, N_{rs}] = -\delta_{js}\tilde{L}_{ir} + \epsilon\delta_{is}\tilde{L}_{jr} \tag{54k}$$

$$[L_{ij}, \tilde{N}_{rs}] = \delta_{js}\tilde{L}_{ir} + \epsilon\delta_{is}\tilde{L}_{jr} \tag{54l}$$

$$[L_{ij}, \tilde{M}_{rs}] = -\epsilon\delta_{jr}\tilde{N}_{is} + \delta_{js}\tilde{N}_{ir} - \delta_{ir}\tilde{N}_{js} + \epsilon\delta_{is}\tilde{N}_{jr} \tag{54m}$$

$$[M_{ij}, \tilde{N}_{rs}] = \delta_{jr}\tilde{M}_{is} + \epsilon\delta_{ir}\tilde{M}_{js} \tag{54n}$$

$$[\tilde{L}_{ij}, M_{rs}] = -\epsilon\delta_{jr}\tilde{N}_{is} - \delta_{js}\tilde{N}_{ir} + \delta_{ir}\tilde{N}_{js} + \epsilon\delta_{is}\tilde{N}_{jr} \tag{54p}$$

$$[\tilde{M}_{ij}, \tilde{M}_{rs}] = [\tilde{L}_{ij}, \tilde{L}_{rs}] = [M_{ij}, \tilde{M}_{rs}] = [L_{ij}, \tilde{L}_{rs}] = 0. \tag{54q}$$

The relations (54*a*–*e*) form the Lie algebras of either symplectic Sp($2p$) (for $\epsilon = +$) or orthogonal SO($2p$) (for $\epsilon = -$) groups. The relations (54*f*–*q*) fill up these algebras to the Lie algebra of the unitary SU($2p$) group ($p \to \infty$). The extension of this group to SU($2p+1$) (of course for $\epsilon = -$ only) gives a possibility of including the operators b_r, b_r^+, c_r, c_r^+ themselves into the Lie algebra of the group (Govorkov 1978).

4. Conclusion

The above consideration showed that the principle of indistinguishability of identical particles brings us to the paraquantisation schemes rather than to the ordinary Bose- or Fermi-quantisation. The naturalness of the schemes of para-Bose-, para-Fermi- and uniquantisation lies in the existence of three simple Lie algebras in even-dimensional spaces: Lie algebras of symplectic, orthogonal and unitary groups, respectively. Thus, the Lie algebraic approach to the field quantisation developed by Kamefuchi and Takahashi (1962), Ryan and Sudarshan (1963), Geyer (1968), Bracken and Green (1973) and Palev (1974) is not accidental but comes from the indistinguishability of identical particles.

On the other hand, it is well known that the parafield can be presented in the framework of the so-called 'Green ansatz' (Green 1953, Greenberg and Messiah 1965) as sums of a certain number of ordinary fermion or boson fields with anomalous (opposite) mutual commutation relations. The latter does not really matter inasmuch as the anomalous mutual commutation relations can be changed into the normal ones by means of the convenient Klein transformation (Govorkov 1966, Drühl *et al* 1970). Thus, the parafield can be reduced to the conglomerate of usual canonical fields degenerated in some hidden degree of freedom. (For some peculiarities of this representation for the uniquantisation see Govorkov (1978).) Hence, the introduction of hidden degrees of freedom in such a manner is the only generalisation of usual statistics when the field quantisation is restricted to the simple Lie algebraic schemes. Moreover, the canonical Bose–Einstein and Fermi–Dirac statistics can be considered as special but fundamental representations of Green's paracommutation relations. The possibility of introducing physical internal symmetries of the type of isospin, SU(3) or higher hadronic symmetries as well as some leptonic ones through the para- or uniquantisation has only started to be investigated (Govorkov 1968, 1969, 1973, 1978, Green 1972, Bracken and Green 1973, Ohnuki and Kamefuchi 1973a, b).

Our consideration showed only the sufficiency of the above-mentioned schemes of generalised quantisation but not their necessity. It is still an open question: could any more complicated schemes founded on non-simple Lie algebras exist? The main result of this paper asserts that all of them must include the Lie algebra of the SU($p \to \infty$) group as their subalgebra.

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